On Locality-Sensitive Orderings & Their Applications

Timothy Chan, Sariel Har-Peled, Mitchell Jones
ITCS '19, January 10-12, 2019

University of Illinois at Urbana-Champaign
Low dimension proximity problems: $d = O(1)$

- Nearest neighbor
  - [Indyk, Motwani ’98]
  - [Liao et al. ’01]
  - [Chan ’02]

- Closest pair problems
  - [Eppstein ’95]
  $\implies \approx O(\log^3 n)$

- Spanners/MST
  - [Roditty ’12]
  - [Gottlieb, Roditty ’08]

**Goal:** Design dynamic data structures which return a $(1 + \varepsilon)$-approximation
Ordering of points
Ordering of points
Locality-sensitive orderings
Locality-sensitive orderings
Definition: Locality-Sensitive Orderings

Let $\varepsilon \in (0, 1)$. A collection of orderings $\Pi$ over $[0, 1)^d$ s.t. for all $p, q \in [0, 1)^d$, there exists a $\sigma \in \Pi$ where:

$$\forall p \prec_\sigma z \prec_\sigma q : \min(||z - p||, ||z - q||) \leq \varepsilon \|p - q\|.$$
**Definition: Locality-Sensitive Orderings**

Let $\varepsilon \in (0, 1)$. A collection of orderings $\Pi$ over $[0, 1)^d$ s.t. for all $p, q \in [0, 1)^d$, there exists a $\sigma \in \Pi$ where:

$$\forall p \prec_\sigma z \prec_\sigma q : \min(\|z - p\|, \|z - q\|) \leq \varepsilon \|p - q\|.$$ 

**Theorem**

There are locality-sensitive orderings of size $O\left((1/\varepsilon^d) \log(1/\varepsilon)\right)$. 
Problem

Maintain a pair \((r', b')\) s.t. \(\|r' - b'\| \leq (1 + \varepsilon) \cdot \min_{(r,b)} \| r - b \|\).
Application: Bichromatic closest pair

- Idea: Solve the 1D problem
Application: Bichromatic closest pair

- Idea: Solve the 1D problem
- Maintain order in a binary tree
Application: Bichromatic closest pair

- Idea: Solve the 1D problem
- Maintain order in a binary tree
- Maintain min-heap of consecutive red/blue pairs
Application: Bichromatic closest pair

- Idea: Solve the 1D problem
- Maintain order in a binary tree
- Maintain min-heap of consecutive red/blue pairs
- Easily made dynamic
Application: Bichromatic closest pair
Application: Bichromatic closest pair
Application: Bichromatic closest pair
Application: Bichromatic closest pair
Application: Bichromatic closest pair

\[ \|r' - b'\| \leq \|r' - r\| + \|r - b\| + \|b - b'\| \]

\[ \leq (1 + 2\varepsilon) \|r - b\| \]
1. Bichromatic closest pair (improved time to $O(\log n)$)
1. Bichromatic closest pair (improved time to $O(\log n)$)
2. Dynamic spanners (simpler, [Gottlieb, Roditty ’08])
Applications

1. Bichromatic closest pair (improved time to $O(\log n)$)
2. Dynamic spanners (simpler, [Gottlieb, Roditty ’08])
3. Static vertex-fault-tolerant spanners (simpler, [Kapoor, Li ’13] [Solomon ’14])
Applications

1. Bichromatic closest pair (improved time to $O(\log n)$)
2. Dynamic spanners (simpler, [Gottlieb, Roditty ’08])
3. Static vertex-fault-tolerant spanners (simpler, [Kapoor, Li ’13] [Solomon ’14])
4. Dynamic vertex-fault-tolerant spanners (new)
1. **Bichromatic closest pair** (improved time to $O(\log n)$)

2. **Dynamic spanners** (simpler, [Gottlieb, Roditty ’08])

3. **Static vertex-fault-tolerant spanners** (simpler, [Kapoor, Li ’13] [Solomon ’14])

4. **Dynamic vertex-fault-tolerant spanners** (new)

5. **Approximate nearest neighbor** (not new, [Chan ’02])
Timothy M. Chan.
**Closest-point problems simplified on the RAM.**

T.-H. Hubert Chan, Mingfei Li, Li Ning, and Shay Solomon.
**New doubling spanners: Better and simpler.**

David Eppstein.
**Dynamic Euclidean minimum spanning trees and extrema of binary functions.**
Lee-Ad Gottlieb and Liam Roditty. 
**An optimal dynamic spanner for doubling metric spaces.** 

Piotr Indyk and Rajeev Motwani. 
**Approximate nearest neighbors: Towards removing the curse of dimensionality.** 

Sanjiv Kapoor and Xiang-Yang Li. 
**Efficient construction of spanners in $d$-dimensions.** 